Laminar Flow of Power Law Fluids in Concentric Annuli

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Ranger (1994) demonstrated recently that it is possible to achieve higher volumetric flow rate for the laminar flow of incompressible Newtonian fluids in a hollow annular pipe than that in an annulus with the solid core for equal flow areas and pressure gradients. His calculations reveal enhancements of up to 250% in flow rate for certain combinations of the sizes of pipes involved in it. Aside from the theoretical interest in this problem (e.g., Shivakumar and Ji, 1993), this result is of considerable pragmatic importance when different fluids need to be transported between the same points. However, no physical explanation was presented to explain this phenomenon.

This note demonstrates that greater enhancements in flow rate are possible for the laminar flow of incompressible power law fluids under the same circumstances. Representative numerical results are presented for a range of combinations of the flow behavior index and geometric parameters to elucidate the interplay between the nonlinear fluid characteristics and the geometrical parameters. The observed trends are qualitatively explained by considering limiting cases and in terms of the results for known geometries.

Analysis

Consider the two flow geometries shown in Figure 1. The expressions for the volumetric flow rate for the laminar flow of power law fluids through a circular tube and through a concentric annulus are well documented in the literature (e.g., Bird et al., 1987). For the flow arrangement A, the total volumetric flow rate is simply the sum of the individual flow rates through the inner circular region ($0 \le r \le R_2$), say Q_1, and through the annular area ($R_3 \le r \le R_4$), say Q_2; the corresponding relations are

$$Q_1 = \alpha(s) R_2^{s+3} \tag{1}$$

$$Q_2 = \alpha(s) R_4^{s+3}\left[(1-\beta_{34}^2)^{s+1} - k_{34}^{1-s}(\beta_{34}^2 - k_{34}^2)^{s+1}\right] \tag{2}$$

where

$$s = 1/n \quad \text{and} \quad \alpha(s) = \frac{\pi}{s+3}\left(\frac{-\Delta p}{2mL}\right)^s \tag{3}$$

The total volumetric flow rate, $Q_A = Q_1 + Q_2$.

Similarly, for the flow geometry B, one can rewrite Eq. 2 to obtain the volumetric flow rate Q_B (m^3/s) as

$$Q_B = \alpha(s) R_4^{s+3}\left[(1-\beta_4^2)^{s+1} - k_4^{1-s}(\beta_4^2 - k_4^2)^{s+1}\right] \tag{4}$$

Note that the unknown radius R (m) in configuration B has been eliminated by satisfying the requirement of equal flow areas, i.e.,

$$\pi(R_4^2 - R_3^2) + \pi R_2^2 = \pi(R_4^2 - R^2) \tag{5}$$

or

$$R^2 = R_3^2 - R_2^2 \tag{6}$$

Furthermore, various pipe radii have been normalized with respect to R_4; $k_{24} = R_2/R_4$; $k_{34} = R_3/R_4$, etc. Clearly, Eq. 6 suggests

$$k_4^2 = k_{34}^2 - k_{24}^2 \tag{7}$$

Finally, the parameters β_{34} and β_4 are only dependent on the power law index and geometric ratios k_{34}, k_4, etc., and their values have been tabulated by Bird et al. (1987).

The enhancement in flow rate S is defined as

$$S = \frac{Q_A}{Q_B} = f(k_{24}, k_{34}, k_4, n) \tag{8}$$

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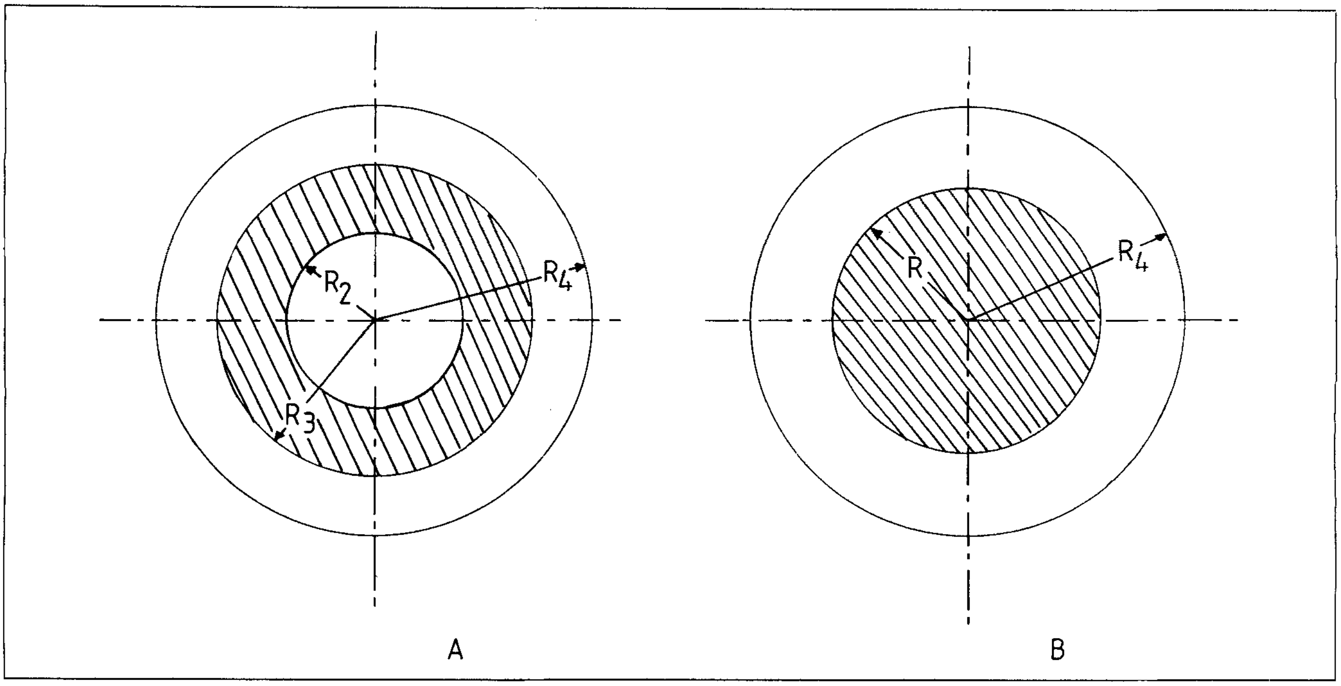

Figure 1. Representation of flow configuration.

and one can thus estimate the values of S for known values of the ratios of various radii and the power law index n. However, it is convenient to set $R_4 = 1$ and to introduce the following definitions to be consistent with the form of presentation that used by Ranger (1994)

$$\frac{R_2}{R_4} = k_{24} = 1 - \epsilon - \delta \tag{9}$$

$$\frac{R_3}{R_4} = k_{34} = 1 - \epsilon \tag{10}$$

where $0 \leq \epsilon \leq \epsilon + \delta \leq 1$.

In this work, the value of S has been computed for $0.1 \leq n \leq 1$; $0.1 \leq \epsilon \leq 0.6$ and $0.1 \leq \delta \leq 0.4$ thereby encompassing wide ranges of non-Newtonian behavior and geometric parameters.

Results and Discussion

At the outset, the flow enhancement factor S was evaluated for Newtonian fluids ($n = 1$) in the range $0 \leq \bullet \leq 0.6$ and $0 \leq \delta \leq 0.4$. The present values agreed fully with those of Ranger (1994) in the complete range of parameters, except for $\delta = 0.4$ in which case the two values differed by about 0.5%. This discrepancy is mainly due to the fact that the values of β tabulated by Bird et al. (1987) are based on the more extensive numerical results of Hanks and Larsen (1979) and perhaps thus entail some interpolation errors. This minor discrepancy, however, does not change the overall pattern of the results for Newtonian or power law fluids.

Figures 2 and 3 show the representative results on the variation of S with the flow behavior index and geometric parameters. As mentioned earlier, even greater values of S (up to 1,000!) result for power law fluids. Note increasing enhancements in flow in a two flow region configuration with the increasing value of δ (i.e., decreasing inner region) and with the increasing degree of non-Newtonian behavior. On the other hand, Figure 3 shows that for smaller outer flow region (in case A), the flow rate goes through a maximum ca. $\delta \sim 0.2$–0.3, the exact location being slightly dependent on the value of the power law index. However, for $\epsilon > 0.3$, the

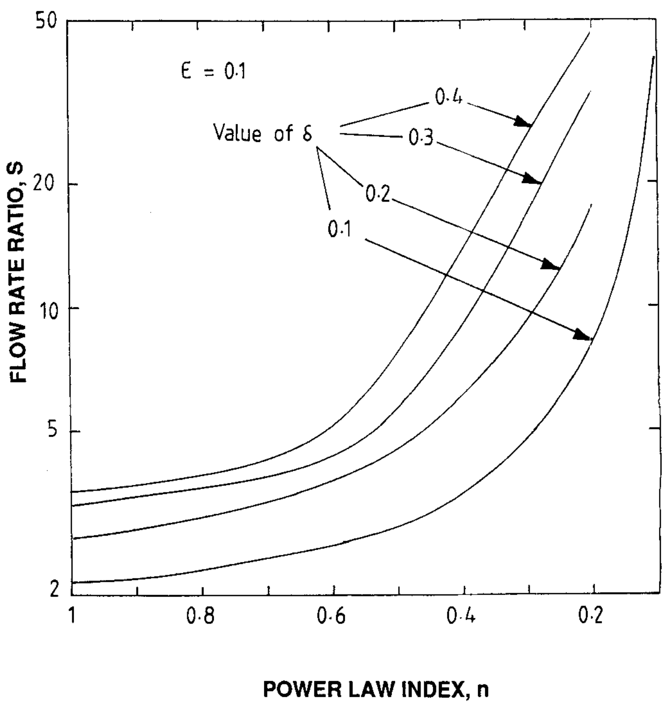

Figure 2. Dependence of S on δ and n for • = 0.1.

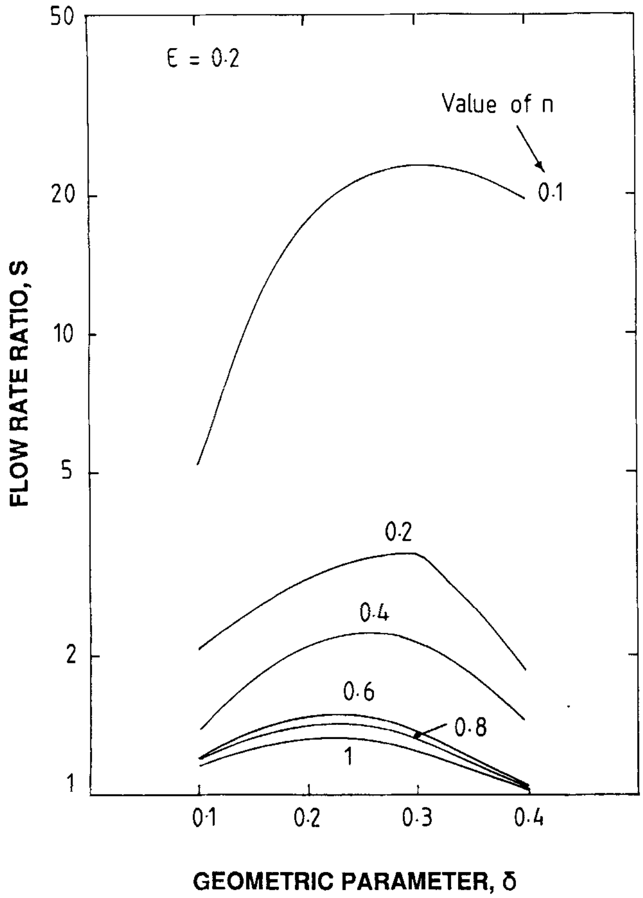

Figure 3. Dependence of S on n and δ for $\epsilon = 0.2$.

value of S is always less than unity irrespective of the value of δ and/or n thereby suggesting that there is no advantage in using a two flow region configuration over that a single annulus. The flow enhancement in the case of Newtonian fluids can be attributed to the fact that the arrangement A has greater wetted surface area, and thus the resulting shear stress is lower than that in B at the same value of the imposed pressure gradient. This effect is further augmented for power law fluids due to their shear dependent viscosity. Clearly, the fluid is subjected to higher shear rates in the two flow regions arrangement A than that in the single annulus as in B. The shear rate is strongly dependent on the choice of geometrical parameters, namely, ϵ and δ. The effective viscosity in turn depends upon the value of n. This partly explains the type of dependence of S on n as seen in Figures 2 and 3. More physical insights into the nature of flow and the resulting dependencies of S on the geometric and power law parameters can be gained by considering the following idealized flow geometries. At the outset, let us consider the flow of Newtonian fluids with the limitations $\epsilon \to 0$: $k_{24} \sim 1 - \delta$ and $\delta \geq 0.5$. For these conditions, the flow rate in arrangement A can be written as

$$Q_A = \frac{\pi R_2^4}{8\mu}\left(-\frac{\Delta p}{L}\right) \tag{11}$$

While the flow in arrangement B can be envisioned as one in a plane slit of thickness say, $2H$ and width $W = 2\pi$. The flow rate under these conditions is given by

$$Q_B = \frac{2HW}{3\mu} H^2 \left(-\frac{\Delta p}{L}\right) \tag{12}$$

From the requirement of equal flow areas in two cases, it can be readily seen that

$$H = \frac{\pi R_2^2}{2W} = \frac{(1-\delta)^2}{4} \tag{13}$$

and therefore, the ratio S becomes

$$S = \frac{Q_A}{Q_B} = \frac{6}{(1-\delta)^2} \tag{14}$$

Evidently, S increases with the increasing value of δ which is consistent with the values reported by Ranger (1994) and obtained in this work for $\epsilon = 0.1$. Smaller the value of ϵ, the closer are predictions of Eq. 14 to the actual values of S.

The corresponding expression for power law liquids is obtained as

$$S = \frac{4^{(s+2)/2}(s+2)}{(s+3)(1-\delta)^{s+1}} \tag{15}$$

For fixed values of δ, Eq. 15 predicts an increase in the value of S with the increasing extent of shear thinning behavior (i.e., decreasing n) which is in line at least qualitatively with the trends seen in Figure 2. Furthermore, the smaller the value of δ, greater is the value of S for a fixed value of n. Evidently, if the value of $\delta < \sim 0.5$, one can no longer neglect the curvature effects in case B and hence the expression for the flow in an annulus must be used.

Some further progress can be made by considering the flow mechanism from another vantage point. The flow configuration A can be visualized as two pipes of radii R_2 and $R_4 - R_3$ respectively arranged in parallel, i.e.,

$$R_2 = 1 - \epsilon - \delta \quad \text{and} \quad R_4 - R_3 = 1 - \epsilon \tag{16}$$

Likewise the flow configuration B can be seen as a circular pipe of radius $(R_4 - R)$ where the unknown radius R is related to R_2 and R_3 via Eq. 6, and is given by $\{(2-2\epsilon-\delta)(\delta)\}^{1/2}$. Now recalling that for a Newtonian fluid, $Q \alpha R^4$, the ratio S becomes

$$S = \frac{R_2^4 + (R_4 - R_3)^4}{(R_4 - R)^4} = \frac{(1-\epsilon-\delta)^4 + \epsilon^4}{\left(1 - \sqrt{\delta(2-2\epsilon-\delta)}\right)^4} \tag{17}$$

Equation 17 predicts both flow enhancements ($S > 1$) as well as flow reductions ($S < 1$) for suitable choices of ϵ and δ. More importantly, it can be readily shown that for each value of ϵ, S attains a maximum value at a certain value of δ. Obviously, quantitative agreement between the predictions of Eq. 17 and the actual values of S was reported by Ranger

(1994). Herein, this cannot be expected, but the trends displayed by Eq. 17 are certainly in qualitative agreement over the complete ranges of ϵ and δ. The aforementioned reasoning can be easily extended to power law fluids by noting that $Q \propto R^{s+3}$, the value of S is given by

$$S = \frac{(1-\epsilon-\delta)^{s+3}+\epsilon^{s+3}}{\left(1-\sqrt{\delta(2-2\epsilon-\delta)}\right)^{s+3}} \tag{18}$$

The predictions of Eq. 18 also support the trends displayed by the results shown in Figures 2 and 3, as well as by the results not shown here in the range $0.1 \leq n \leq 1$.

The considerations of the foregoing limiting and special cases clearly demonstrate the intricate interplay between the geometric parameters and ϵ and δ and fluid flow thereby resulting in the regions of flow enhancement and flow reductions for Newtonian fluids, depending upon the ratio of equivalent flow areas rather than the actual flow areas. This ratio changes with the values of ϵ and δ. This interplay is further accentuated in the case of power law fluids owing to the shear dependent viscosity.

Conclusion

In this work, it is demonstrated that with a suitable choice of geometric parameters, it is possible to achieve greater flow rates through concentric pipes than that in a single flow region configuration for equal cross-sectional areas and the same imposed pressure gradient. Conversely, it is possible to effect reduction in a pumping energy requirement by using a multiple flow region pipe instead of a single concentric annulus. The flow enhancements as large as 1,000 can be achieved with suitable choices of geometric parameters for extremely shear thinning materials such as china clay suspensions (Farooqi and Richardson, 1980). However, maximum enhancements in flow are obtained for $\epsilon \rightarrow 0$ and $\delta \rightarrow 1$. We conclude by offering some physical insights into the nature of this complex phenomenon in terms of well-known simple flows.

Notation

m = power law consistency index, $\text{Pa}\cdot\text{s}^n$
$(-\Delta p/L)$ = pressure gradient, Pa/m
s = reciprocal of flow behavior index
α = function of n, m, L, $-\Delta p$, Eq. 3, $\text{m}^{-1/n}/\text{s}$
$\beta_4, \beta_{24}, \beta_{34}$ = functions of radii ratio and n
ϵ = geometric parameter

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